UNIVERSITI TEKNOLOGI MALAYSIA

SME 3023 Applied Numerical Methods

Solution of Simultaneous Linear Equations Part 2

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Outline

1. Methods Available
   - Gauss Elimination
   - Gaussian-Jordan Elimination
   - LU Decomposition
   - Jacobi's Iteration Method
   - Gauss-Siedel Iteration Method

Outline

Direct (elimination) Methods

1. Gauss Elimination Method
2. Gauss-Jordan Method
3. LU Decomposition Method

Indirect (iterative) Methods

1. Jacobi's Method
2. Gauss-Siedel Method
3. Relaxation Method

- The goal is to construct an equivalent upper-triangular system $Ux = b$ that can be solved by the method of upper-triangular linear systems
- Two linear systems of dimension $n \times n$ are said to be *equivalent* provided that their solution sets are the same.
- When certain elementary operations are applied to a given set of system, the solution sets do not change
- **Elementary Operations:**
  The following operations applied to a linear system yield an equivalent system:

Operation	**Description**
Interchanges	The order of two equations can be changed
Scaling	Multiplying an equation by a non-zero constant
Replacement	An equation can be replaced by the sum of itself and a nonzero multiple of any other equation

- **Elementary Row Operations on Augmented Matrix:**
  The following operations applied to an augmented matrix yield an equivalent linear system:

Operation	**Description**
Interchanges	The order of two rows can be changed
Scaling	Multiplying an equation by a non-zero constant
Replacement	The row can be replaced by the sum of itself and a nonzero multiple of any other row; i.e. $\text{row}_r = \text{row}_r - m_{rp} \times \text{row}_p$

- The number a_{rr} in coefficient matrix A that is used to eliminate a_{kr}, where $k = r+1, r+2, \ldots, n$ is the rth *pivotal element*, and the rth row is called the *pivot row*

- Consider the augmented matrix $[C]$ of a system of equations $[A]\vec{x} = \vec{b}$

$$[C] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2,n+1} \\ \dots & & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n,n+1} \end{bmatrix}$$

  where the first n columns of $[C]$ are same as the columns of $[A]$ and the last $(n+1)^{\text{st}}$ column is the same as the constant vector $\vec{b}$.
- Using $(n-1)$ elimination steps, $[C]$ is reduced to an upper triangular augmented matrix $[C^{n-1}]$ of the form

$$[C^{(n-1)}] = \begin{bmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \dots & a_{1,n-1}^{(0)} & a_{1n}^{(0)} & a_{1,n+1}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \dots & a_{2,n-1}^{(1)} & a_{2n}^{(1)} & a_{2,n+1}^{(1)} \\ \dots & & & & & & \\ 0 & 0 & 0 & \dots & 0 & a_{nn}^{(n-1)} & a_{n,n+1}^{(n-1)} \end{bmatrix} \quad (1)$$

  where superscript indicates elimination step to obtain a particular coefficient.

- The first elimination step is applied to obtain elements $a_{ij}^{(1)}$ as

$$a_{ij}^{(1)} = a_{ij}^{(0)} - \frac{a_{i1}^{(0)}}{a_{11}^{(0)}} a_{1j}^{(0)}; \qquad \begin{cases} i = 2, 3, \ldots, n; \\ j = 1, 2, 3, \ldots, n+1 \end{cases} \tag{2}$$

  with the first row used as the pivot row and a_{11} the pivot element.
  The general elimination formula is

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}; \qquad \begin{cases} i = k+1, k+2, \ldots, n \\ j = k, k+1, \ldots, n+1 \\ k = 1, 2, 3, \ldots, n-1 \end{cases} \tag{3}$$

  pivot element is $a_{kk}^{(k-1)}$, which cannot be zero!!
- At the end of $(n-1)$ elimination steps, the system of equations will be reduced to a triangular form of Eq. (1).

- Using back substitution, the unknowns x_i are computed as

$$x_n = \frac{a_{n,n+1}}{a_{nn}} \tag{4}$$

$$x_i = \frac{a_{i,n+1} - \sum\limits_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}; \qquad i = n-1, n-2, \ldots, 1 \tag{5}$$

Example: In a three-equation system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

elimination steps involved are

1. Multiply row 1 by the factor $f_{21} = \frac{a_{21}}{a_{11}}$ and subtract the result from row 2 to eliminate a_{21}, $a_{22} \longrightarrow a'_{22}$, $a_{23} \longrightarrow a'_{23}$
2. Multiply row 1 by the factor $f_{31} = \frac{a_{31}}{a_{11}}$ and subtract the result from row 3 to eliminate a_{31}, $a_{32} \longrightarrow a'_{32}$, $a_{33} \longrightarrow a'_{33}$
3. Multiply row 2 by the factor $f_{32} = \frac{a'_{32}}{a'_{22}}$ and subtract the result from row 3 to eliminate a_{32}, $a_{33} \longrightarrow a''_{33}$

The upper triangulation of $[A]$ above created zeros in a_{21}, a_{31} and a_{32} to give us

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

Problem Statement:

Reduce the following coefficient matrix $[A]$ to an upper triangular matrix suitable for backwards substitution

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

Solution:

Start with

$$A = \begin{array}{c c} \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix} & \begin{array}{c} R1 \\ R2 \\ R3 \end{array} \end{array}$$

Solution (continued):
Use $R1$ to eliminate elements of $[A]$ in $C1$ below a_{11}

$$f_{21} = \frac{a_{21}}{a_{11}} = \frac{-2}{4} = -0.50 \qquad f_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4} = +0.25$$

and apply the following two operations to $R1$ and $R2$, respectively

$$R2' = R2 - f_{21}R1 \qquad R3' = R3 - f_{31}R1$$

to get

$$\begin{array}{cl} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.50 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} & \begin{array}{l} R1 \\ R2 \rightarrow R2' \\ R3 \rightarrow R3' \end{array} \end{array}$$

Solution (continued):

Use $R2'$ to eliminate elements of $[A]$ in $C2$ below a'_{22}

$$f_{32} = \frac{a'_{32}}{a'_{22}} = \frac{1.25}{-2.5} = -0.50$$

and apply the following operation to $R3'$

$$R3'' = R3' - f_{32}R2'$$

to finally get

$$U = \begin{array}{cl} \left[\begin{array}{crr} 4 & 3 & -1 \\ 0 & -2.50 & 4.5 \\ 0 & 0 & 8.5 \end{array}\right] & \begin{array}{l} R1 \\ R2' \\ R3' \to R3'' \end{array} \end{array}$$

Problem Statement:

Reduce the following set of equations to a triangular form:

$$2x_1 - x_2 + x_3 = 4 \tag{E1}$$

$$4x_1 + 3x_2 - x_3 = 6 \tag{E2}$$

$$3x_1 + 2x_2 + 2x_3 = 15 \tag{E3}$$

Solution:

Work through the example.

Problem Statement:
Find the solution of the following set of equations using the Gauss elimination method:

$$2x_1 - x_2 + x_3 = 4 \qquad \text{(E1)}$$
$$4x_1 + 3x_2 - x_3 = 6 \qquad \text{(E2)}$$
$$3x_1 + 2x_2 + 2x_3 = 15 \qquad \text{(E3)}$$

Solution:
Work through the example.

Problem Statement:
Express the following system in augmented matrix form and find an equivalent upper-triangular system and the solution.

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}
$$

Solution:
Work through the example.

Problem Statement:

Modify `gauss.f` to use Gauss elimination method to solve

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}
$$

Re-write the code in `gauss.f` in the form of a subroutine.

- This is an extension of Gauss elimination.
- It uses the same elementary operations as Gauss elimination
- The difference between the two:
  - Gauss elimination reduces all the off-diagonal elements below the diagonal to zero
  - Gauss-Jordan elimination reduces all the off-diagonal elements both, below and above the diagonal to zero. For a 3×3 matrix, for instance, we will end up with

$$[D] = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}$$

- Involves factoring coefficient matrix into two triangular matrices that can be used to efficiently evaluate unknown vector $\{x\}$ for different RHS constant vector $\{b\}$
- For system of linear algebraic equations $[A]\{x\} = \{b\}$ Gauss elimination becomes very inefficient when solving equations with the same coefficients $[A]$ but with different RHS constants $\{b\}$, for example

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}
\qquad
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 3 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 8 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 2 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 7
\end{aligned}
$$

- A system of linear algebraic equations $[A]\{x\} = \{b\}$ re-arranged

$$[A]\{x\} - \{b\} = 0 \tag{6}$$

- Suppose that Eq. (6) could be expressed as upper triangular system, say for 3×3 system,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} \tag{7}$$

and then re-arranged to be

$$[U]\{x\} - \{d\} = 0 \tag{8}$$

- Now assume that there is a lower triangular system, with 1's on the diagonal

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \tag{9}$$

that has the property that when Eq. (8) is premultiplied by it will result in Eq. (6)

$$[L]([U]\{x\} - \{d\}) = [A]\{x\} - \{b\} \tag{10}$$

- If Eq. (10) holds, it follows from rules of matrix multiplication that

$$[L][U] = [A] \tag{11}$$

and

$$[L]\{d\} = \{b\} \tag{12}$$

- Take a three-equation system for *LU* decomposition

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

- Steps
  1. Multiply row 1 by the factor $f_{21} = \frac{a_{21}}{a_{11}}$ and subtract the result from row 2 to eliminate a_{21}, $a_{22} \rightarrow a'_{22}$, $a_{23} \rightarrow a'_{23}$
  2. Multiply row 1 by the factor $f_{31} = \frac{a_{31}}{a_{11}}$ and subtract the result from row 3 to eliminate a_{31}, $a_{32} \rightarrow a'_{32}$, $a_{33} \rightarrow a'_{33}$
  3. Multiply row 2 by the factor $f_{32} = \frac{a'_{32}}{a'_{22}}$ and subtract the result from row 3 to eliminate a_{32}, $a_{33} \rightarrow a''_{33}$

- Triangulation of $[A]$ above would have created zeros in a_{21}, a_{31} and a_{32}, yielding

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

- If, instead of storing zeros in a_{21}, a_{31} and a_{32}, we store f_{21}, f_{31} and f_{32}, we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ f_{21} & a'_{22} & a'_{23} \\ f_{31} & f_{32} & a''_{33} \end{bmatrix} \tag{13}$$

- Eq. (13) represent an efficient storage of the $[LU]$ decomposition of $[A]$, i.e.

$$[A] \longrightarrow [L][U] \tag{14}$$

where

$$[U] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \qquad [L] = \begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

Problem Statement:
Compute the LU decomposition matrix $[A]$ where

$$A = \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix}$$

Solution:
Start with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ -2 & -4 & 5 \\ 1 & 2 & 6 \end{bmatrix} \begin{array}{l} R1 \\ R2 \\ R3 \end{array}$$

Solution (continued):

Use $R1$ to eliminate elements of $[A]$ in $C1$ below a_{11}

$$f_{21} = \frac{a_{21}}{a_{11}} = \frac{-2}{4} = -0.50 \qquad f_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{4} = +0.25$$

and apply the following two operations to $R1$ and $R2$, respectively

$$R2' = R2 - f_{21}R1 \qquad R3' = R3 - f_{31}R1$$

to get

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 \\ -0.50 & 1 & 0 \\ 0.25 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.50 & 4.5 \\ 0 & 1.25 & 6.25 \end{bmatrix} & \begin{matrix} R1 \\ R2' \\ R3' \end{matrix} \end{array}$$

Solution (continued):
Use $R2'$ to eliminate elements of $[A]$ in $C2$ below a'_{22}

$$f_{32} = \frac{a'_{32}}{a'_{22}} = \frac{1.25}{-2.5} = -0.50$$

and apply the following operation to $R3'$

$$R3'' = R3' - f_{32}R2'$$

to get

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.50 & 1 & 0 \\ 0.25 & -0.50 & 1 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 4 & 3 & -1 \\ 0 & -2.50 & 4.5 \\ 0 & 0 & 8.5 \end{bmatrix} \end{array} \begin{array}{l} R1 \\ R2' \\ R3'' \end{array}$$

Problem Statement:

Use Matlab to compute the *LU* decomposition and find the solution for the linear system

$$\begin{bmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 7.85 \\ -19.3 \\ 71.4 \end{Bmatrix}$$

Solution:

Matlab Session

```
>> A = [3 -0.1 -0.2; 0.1 7 -0.3; 0.3 -0.2 10];
>> b = [7.85 -19.3 71.4]';
>> [L U p] = lu(A)          % decompose A into L and U
>> x = inv(A)*b             % solve for x
```

- Consider a system of simultaneous linear equations with there equations and three unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \tag{15a}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \tag{15b}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \tag{15c}$$

- Eqs. (15) in matrix form

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{[A]} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{\{x\}} = \underbrace{\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}}_{\{b\}} \tag{16}$$

or simply

$$[A]\{x\}^{(k)} = \{b\} \tag{17}$$

- From Eqs. (15),

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} \tag{18a}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k)} - \frac{a_{23}}{a_{22}} x_3^{(k)} \tag{18b}$$

$$x_3^{(k+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(k)} - \frac{a_{32}}{a_{33}} x_2^{(k)} \tag{18c}$$

If we initialize the vector of unknowns $\{x\}^{(k)}$ with

$$\{x\}^{(k)} = \begin{Bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{Bmatrix} \tag{19}$$

and substitute into Eq. (18), a new vector $\{x\}^{(k+1)}$ is calculated. This process is continued until sufficient accuracy is achieved, i.e. until

$$\{x\}^{(k+1)} \cong \{x\}^{(k)} \qquad k = 0, \ldots, N \tag{20}$$

where N is the total number of iteration performed.

- In general, for $n \times n$ set, Eqs. (18) can be expressed in the following form:

$$\{x\}^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k)} \right) \tag{21}$$

or in matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}^{(k+1)} = \begin{Bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{Bmatrix} - \begin{bmatrix} 0 & A_{12} & \dots & A_{1n} \\ A_{21} & 0 & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}^{(k)} \tag{22}$$

where

$$A_{ij} = \frac{a_{ij}}{a_{ii}}, \quad i = 1, \dots, n, \quad j = 1, \dots, n; \qquad B_i = \frac{b_i}{a_{ii}}, \quad i = 1, \dots, n; \qquad k = 0, \dots, N$$

and N is the total number of iterations.

Problem Statement:

Modify `jacobi.f` to use Jacobi's iteration method to solve

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13\\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28\\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20\\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}$$

Re-write the code in `jacobi.f` in the form of a subroutine.

- This method presupposes that the new solution values are better approximation to the exact solution than the initial assumed values. This is true for most engineering problems. For the 3×3 system in Eqs. (15) the solution for the new values become

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} \tag{23a}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)} \tag{23b}$$

$$x_3^{(k+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(k+1)} - \frac{a_{32}}{a_{33}} x_2^{(k+1)} \tag{23c}$$

- It can be seen in Eqs. (23) that x_1 is computed by assuming x_2 and x_3. Then the new x_1 value and the assumed x_3 value are used to calculate a new x_2 value. This process is continued by substituing the new x_1 and x_2 values to determine a new x_3 value.

- In general

$$x_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)} \tag{24}$$

$$i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad k = 1, \ldots, N$$

- **Example**
  Problem Statement: Modify `ex3s46.f` to use Gauss-Siedel iteration method to solve

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}
$$

  Re-write the code in `ex3s46.f` in the form of a subroutine.

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