UNIVERSITI TEKNOLOGI MALAYSIA

SME 3023 Applied Numerical Methods

Solution of Simultaneous Linear Equations Part 1

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Outline

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Introduction

- Many applications of engineering lead to a system of linear algebraic equations, involving thousands of simultaneous linear equations.
- Recall elementary algebra to represent these linear equations

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned} \tag{1}$$

where coefficient a, $b \ldots f$ are *constants*. The task is to find unknowns x, y.

- Change the notation used in Eq. (1) into a form suitable for programming

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ &\ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{2}$$

where given constants a, b, $\ldots f$ are being replaced by a_{11}, $a_{12} \ldots a_{n,n}$ and the x_1, x_2 $\ldots x_n$ are used instead of x and y.

- System in Eq. (2) can be represented in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_n \end{Bmatrix} \tag{3}$$

or in matrix shorthand as

$$[A]\vec{x} = \vec{b} \tag{4}$$

where $[A]$ is $n \times n$ matrix of coefficients, $\vec{b}$ is a given n-vector, and $\vec{x}$ is unknown solution n-vector to be determined.

Engineering Applications

Samples of engineering applications include

- force analysis of statically determinate structures
- flow of current in electrical circuits
- force analysis of mechanisms
- finite element analysis of heat flow and stress analysis problems
- flow of fluid in pipe network
- least-square curve-fitting calculations.

Problem Statement:
A scaffolding system, consisting of three rigid bars and six wire ropes, is used to support the load P_1, P_2 and P_3 as shown in Figure 1. Find the tensions developed in the ropes A, B, C, D, E and F where $P_1 = 2000\ lb$, $P_2 = 1000\ lb$ and $P_3 = 500\ lb$.

Solution:
Work through the example.

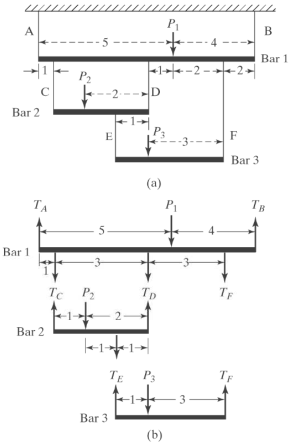

Figure 1 : Scaffolding system.

Problem Statement:
The crank of the four-bar mechanism shown in Figure 2 has an angular velocity of 25 rad/s and angular acceleration of -40 rad/s^2. The crank, coupler, and output links weigh 1.5, 7.7, and 5.8 lb, respectively. A torque T_0 is applied to the crank while maintaining the stated motion, and an external force of 80 lb acts at point P on the coupler as indicated in Figure 2. Find the joint forces and the external torque acting on the crank.

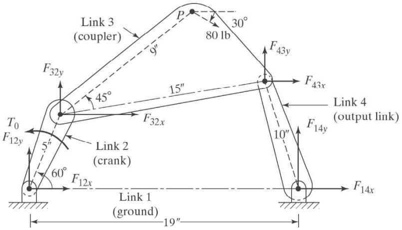

Figure 2 : A four-bar mechanism.

Solution:
Work through the example.

Problem Statement:
Two sides of a square plate of uniform thickness are maintained at 200°C, and the other two sides are exposed to an atmospheric temperature of 0°C, shown in Figure 3. Derive the equations for finding the temperature distribution in the plate.

Solution:
Work through the example.

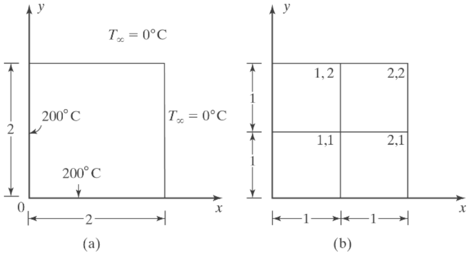

Figure 3 : Temperature distribution in square plate.

Problem Statement:
A mass m is suspended by three cables attached at three points B, C, and D as shown in Figure 4. Let T_1, T_2, T_3 be the tensions in the three cables AB, AC and AD, respectively. If the mass m is stationery, the sum of the tension components in the x, y and z directions must each be zero. Based on this requirement gives derive the simultaneous linear equations for finding the tensions T_1, T_2, T_3 in terms of an unspecified value of the weight mg.

Solution:
Work through the example.

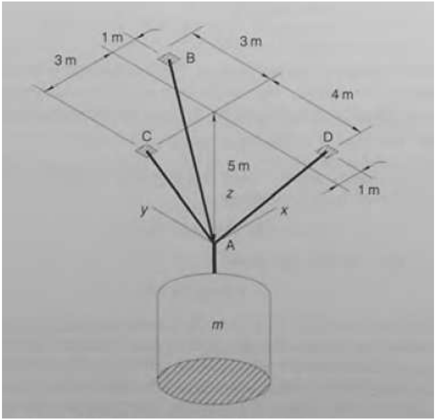

Figure 4 : A mass suspended by three cables.

- A norm or length is a measure of the size of a vector $\vec{x}$, where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

- Euclidean norm or length of this vector is defined as

$$||\vec{x}|| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$$

- In general, L_p-norm of a vector $\vec{x}$ is defined as

$$L_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$$

and if $p \to \infty$

$$L_\infty = \max_i |x_i|$$

Some properties of the vector norms are

- $||\vec{x}|| \geq 0$ for any vector $\vec{x}$
- $||\vec{x}|| = 0$ if and only if $\vec{x} = \vec{0}$
- $||k\vec{x}|| = |k| \, ||\vec{x}||$ for any real or complex number k
- $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$ for any two vectors $\vec{x}$ and $\vec{y}$ of the same order

- A norm or length is a measure of the size of a matrix.
- Useful for defining the condition number of a matrix. Condition number, in turn, used to define degree of ill conditioning of a set of linear equations.
- Matrix norms are defined in terms of its effect on a vector.
- A vector $\vec{x}$ premultiplied by a square matrix A yields a vector $\vec{y}$ that differ from $\vec{x}$. When all combinations of $\vec{x}$ and $\vec{y}$ are considered, the largest ratio of

$$\frac{||\vec{y}||}{||\vec{x}||}$$

is defined as the norm of matrix A

$$||A|| = \max_{\vec{x}=1} \frac{||\vec{y}||}{||\vec{x}||} = \max_{\vec{x}=1} \frac{||A\vec{x}||}{||\vec{x}||}$$

- L_2-norm for matrix A is defined as

$$||A||_2 = \max_{1 \le i \le n} \lambda_i \tag{5}$$

where λ is the ith eigenvalue of matrix A. This L_2-norm is also known as spectral radius or spectral norm of a square matrix A.

- L_e-norm or Euclidean-norm, for $m \times n$ matrix A is defined as

$$||A||_e = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}} \tag{6}$$

and written as

$$||A||_2 \neq ||A||_e \qquad \text{for a matrix, } A$$
$$||\vec{x}||_2 = ||\vec{x}||_e \qquad \text{for a vector, } \vec{x}$$

- Defined using matrix norms as

$$\text{cond}(A) = ||A|| \cdot ||A^{-1}||$$

- For matrix A, $\text{cond}(A) \geq 1.0$
- $\text{cond}(A)$ with a value considerably greater than unity suggests that the system represented by A is ill-conditioned.

- For a square 2×2 matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For a square matrix B (larger than 2×2) the determinant $|B|$ is found by obtaining 2×2 determinants from, as in this case, the 3×3 determinant:

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This is done by crossing out the row and column through a particular element, leaving four elements which form a 2×2 determinant.

$$|B| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- A square $n \times n$ matrix A is singular if it has any one of the following properties:
  - A has no inverse i.e. there is no matrix M such that $AM = MA = I$, where I is the identity matrix.
  - $\det A = |A| = 0$
  - $\text{rank}(A) < n$ (the rank of a matrix is the maximum number of linearly independent rows or columns it contains)
  - $Az = 0$ for some vector $z \neq 0$

- If all constants b_i are zero, Eq. (2) is known as a set of homogeneous (linear) equations.
- A trivial (simple) solution exists and is given by

$$\vec{x} = 0$$

- A nontrivial solution exists only if the determinant of the coefficient matrix A is zero, i.e.

$$|A| = 0$$

- If at least one b_i is non- zero, Eq. (2) is known as a set of nonhomogeneous equations whose solution is made easy by introducing the concept of augmented matrix
- An augmented matrix, C, is defined as the $n \times (n+1)$ matrix obtained by adding the constant vector $\vec{b}$ as the $(n+1)^{\text{st}}$ column to the columns of the matrix A

$$C = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{array}\right]$$

- The augmented matrix is used to check the existence of solution of a system of nonhomogeneous equations
- System of equation in Eq. (2) has a solution if and only if the rank of augmented matrix C is equal to the rank of the coefficient matrix A, i.e.

$$\operatorname{rank}(C) = \operatorname{rank}(A)$$

- Eq. (2) can be solved using Cramer's rule

$$x_i = \frac{|[A_i]|}{|[A]|}; \qquad i = 1, 2, \ldots, n \tag{7}$$

where $[A_i]$ is the matrix obtained by replacing the i^{th} column of matrix $[A]$ with the constant vector $\vec{b}$:

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,i-1} & b_1 & a_{1,i+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2,i-1} & b_2 & a_{2,i+1} & \ldots & a_{2n} \\ \ldots & & & & & & & \\ a_{n1} & a_{n2} & \ldots & a_{n,i-1} & b_n & a_{n,i+1} & \ldots & a_{nn} \end{bmatrix} \tag{8}$$

- Upper triangulation is a quest to find a linear system that is easy to solve. A matrix U is *upper triangular* if all of its entries **below** the main diagonal is zero,

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 + \ldots + u_{1n}x_n &= b_1 \\ u_{22}x_2 + u_{23}x_3 + \ldots + u_{2n}x_n &= b_2 \\ u_{33}x_3 + \ldots + u_{3n}x_n &= b_3 \\ \ldots \\ u_{nn}x_n &= b_n \end{aligned} \tag{9}$$

  If $u_{ii} \neq 0$ for $k = 1, 2, \ldots, n$ then there exists a unique solution to Eq. (9)
- The last equation of system in Eq. (9) involves only x_n, so we solve it first

$$x_n = \frac{b_n}{u_{nn}}$$

  followed successively backward. This backward substitution is

$$x_n = \frac{b_n}{u_{nn}}, \quad x_i = \frac{\left(b_i - \sum_{j=1}^{i-1} u_{ij}x_j\right)}{u_{ii}} \quad i = n, n-1, \ldots, 2$$

- Lower triangulation is another quest to find a linear system that is easy to solve. A matrix L is lower triangular if all of its entries **above** above the main diagonal is zero, i.e. $l_{ij} = 0$ for $i < j$

$$\begin{aligned}
l_{11}x_1 \quad & = b_1 \\
l_{21}x_1 + l_{22}x_2 \quad & = b_2 \\
l_{31}x_1 + l_{32}x_2 + \ldots + l_{3n}x_n \quad & = b_3 \\
\ldots \quad & \\
l_{n1}x_1 + l_{n2}x_2 + l_{n3}x_3 + \ldots + l_{nn}x_n \quad & = b_n
\end{aligned}$$

- For a lower triangular system $Lx = b$, successive substitution is called forward substitution

$$x_1 = \frac{b_1}{l_{11}}, \quad x_i = \frac{\left(b_i - \sum_{j=1}^{i-1} l_{ij}x_j\right)}{l_{ii}} \quad i = 2, \ldots, n$$

Function	Description
rand	randomly generated matrix
det	determinant
norm	norm of vector and matrix
cond	condition number in the 2-norm
rank	rank
inv	inverse
tril	lower triangular part of a matrix
triu	upper triangular part of a matrix

Matlab Session

```
>> A = rand(10,10)
>> det(A)
>> norm(A)
>> cond(A)
>> rank(A)
>> inv(A)
>> tril(A)
>> triu(A)
```

Determine condition number, e.g.

$$\begin{bmatrix} 2 & 1 \\ 1 & 0.5001 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.3 \\ 0.6 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0.4999 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0.3 \\ 0.6 \end{Bmatrix}$$

Matlab Session

```
>> A1 = [2 1; 1 0.5001];
>> A2 = [2 1; 1 0.4999];
>> cond(A1)
ans =
   3.1251e+04
>> cond(A2)
ans =
   3.1250e+04
```

An example of singular matrix

Matlab Session

```
>> A = [1 2 3; 3 4 5; 6 7 8]
>> det(A)
>> norm(A)
>> cond(A)
>> rank(A)
>> inv(A)
>> tril(A)
>> triu(A)
```

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