UNIVERSITI TEKNOLOGI MALAYSIA

SME 3023 Applied Numerical Methods

Solution of Nonlinear Equations

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Sept 2012

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Introduction

- Many engineering analyses require determination of value(s) of variable x that satisfy a nonlinear equation

$$f(x) = 0 \tag{1}$$

  where x is known as roots of Eq. (1) or zeros of function $f(x)$.
- Number of roots maybe finite or infinite depending on nature of problem and physical problem
- Examples of $f(x)$ are

$$\begin{aligned} x^4 - 80x + 120 &= 0 \qquad \textit{polynomial} \\ \tan x - \tanh x &= 0 \qquad \textit{transcendental equation} \end{aligned}$$

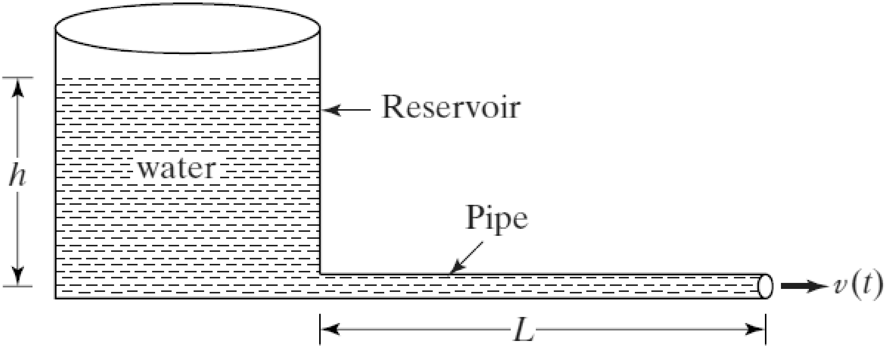

Figure 1 : Discharge of water from reservoir.

Problem Statement:
Water is discharge from a reservoir through a long pipe as shown in Figure 1.By neglecting the change in the level of the reservoir, the transient velocity of the water flowing from pipe, $v(t)$, can be expressed as:

$$\frac{v(t)}{\sqrt{2gh}} = \tanh\left(\frac{t}{2L}\sqrt{2gh}\right)$$

where h is the height of the fluid in the reservoir, L is the length of the pipe, g is the acceleration due to gravity, and t is the time elapsed from the beginning of the flow. Find the value of h necessary for achieving a velocity of $v = 5$ m/s at time $t = 3$ s when $L = 5$ m and $g = 9.81$ m/s^2.

Solution:
Work through the example–see Rao (2002).

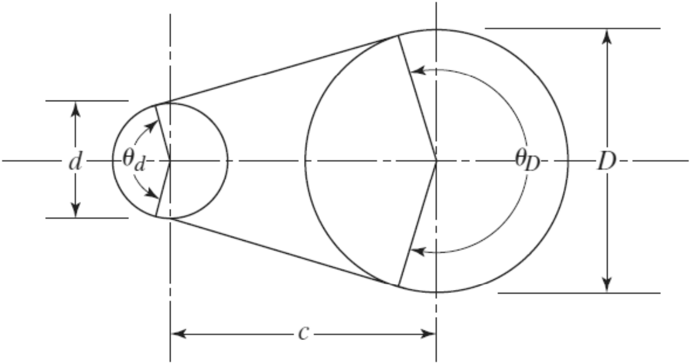

Figure 2 : Open belt drive.

Problem Statement:
The length of a belt in an open-belt drive, L, is given by

$$L = \sqrt{4c^2 - (D-d)^2} + \frac{1}{2}(D\theta_D + d\theta_d) \tag{E1}$$

where

$$\theta_D = \pi + 2\sin^{-1}\left(\frac{D-d}{2c}\right) \qquad \theta_d = \pi - 2\sin^{-1}\left(\frac{D-d}{2c}\right) \tag{E2,E3}$$

c is the centre distance, D is the diameter of the larger pulley, d is the diameter of the smaller pulley, θ_D is the angle of contact of the belt with the larger pulley, and θ_d is the angle of contact of the belt with the smaller pulley, Figure 2. If a belt having a length 11 m is used to connect the two pulleys with diameters 0.4 m and 0.2 m, determine the centre distance between the pulleys.

Solution:
Work through the example–see Rao (2002).

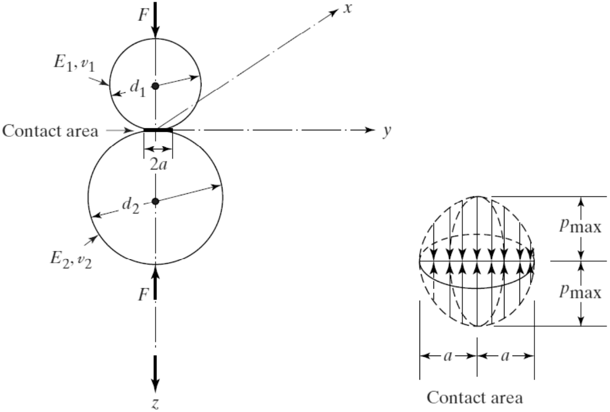

Figure 3 : Contact stress between spheres.

Problem Statement:

The shear stress induced along the z-axis when two spheres are in contact with each other, while carrying a load F, is given by

$$h(\lambda) = \frac{0.75}{1+\lambda^2} + 0.65\lambda \tan^{-1}\left(\frac{1}{\lambda}\right) - 0.65 \tag{E1}$$

where

$$h(\lambda) = \frac{\tau_{zx}}{p_{\max}} \qquad \text{and} \qquad \lambda = \frac{z}{a}$$

in which τ_{zx} is the shear force,

$$p_{\max} = \frac{3F}{2\pi a^2} \tag{E2}$$

is the maximum pressure deveoped at the centre of the contact area, and the radius of the contact area, Figure 3, is

$$a = \left\{ 0.34125F \frac{\left(\frac{1}{E_1} + \frac{1}{E_2} \right)}{\left(\frac{1}{d_1} + \frac{1}{d_2} \right)} \right\}^{1/3} \tag{E3}$$

where E_1 and E_2 are Young's moduli of the two spheres, and d_1 and d_2 are diameters of the two spheres. Poisson's ratios of the two spheres was assume to be 0.3 in deriving Eqs. (E1) and (E3). Determine the value of λ at which the shear stress, given by Eq. (E1), attains its maximum value.

Solution:
Work through the example–see Rao (2002).

Methods Available

- Incremental Search Method
- Bisection Method
- Newton-Raphson Method
- Secant Method
- Regula Falsi Method
- Fixed Point Iteration Method
- Bairstow' Method
- Muller's Method

- Value of x is incremented, by Δx, from an initial value, x_1, successively until a change in the sign of the function $f(x)$ is observed. $f(x)$ changes sign between x_i and x_{i+1}, if it has root in the interval $[x_i, x_{i+1}]$ which implies

$$f(x_i) \times f(x_{i+1}) < 0$$

  wherever a root is crossed.
- Plot of the function is usually very useful in guiding the task of finding the interval.
- A potential problem is the choice of increment length, Δx: too small, the search can be very time-consuming, too great, closely spaced roots might be missed.

Algorithm

1. Sets an initial guess for $x_{i=1}$, and a stepsize Δx.
2. Call the function $f(x)$ to calculate its value at $x_{i=1}$.
3. Increment $x_{i+1} = x_i + \Delta x$.
4. Call the function $f(x)$ to calculate its value at x_{i+1}.
5. Compares the sign of the returned function value $f(x_{i+1})$ to the previous $f(x_i)$.
   - If the sign of $f(x_{i+1})$ does NOT change, repeat from Step 3 again.
   - If the sign of $f(x_{i+1})$ does change, the root lies between x_i and x_{i+1}. Reduce stepsize Δx, set $x_i = x_{i-1}$ and repeat from Step 3.

   Iterate until $|f(x_{i+1}) - f(x_i) \leq \varepsilon|$ where ε is a specified very small number called tolerance.

Problem Statement:
Find the root of the equation

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65 \tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2} = 0 \tag{E1}$$

using the incremental search method with $x_1 = 0.0$ and $\Delta x^{(1)} = 0.1$.

Solution:
Work through the example.

If $f(x)$ is real and continuous in the interval prescribed by a lower bound, x_L and upper bound, x_U and $f(x_L)$ and $f(x_U)$ have opposite signs, such that

$$f(x_L) \times f(x_U) < 0 \tag{2}$$

then there is at least one real root in the interval $[x_L, x_U]$

Algorithm

1. Choose lower x_L and upper x_U guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_L) \times f(x_U) < 0$
2. An estimate of the root x_R is determined by

$$x_R = \frac{x_L + x_U}{2}$$

3. Make the following evaluations to determine in which subinterval the root lies:
   * if $f(x_L) \times f(x_R) = 0$, the root equals x_R. Terminate computation.
   * if $f(x_L) \times f(x_R) < 0$, the root lies in the lower subinterval. Therefore set $x_U = x_R$ and return to step 2
   * if $f(x_L) \times f(x_R) > 0$, the root lies in the upper subinterval. Therefore set $x_L = x_R$ and return to step 2

Iterate until $|f(x_R) \leq \varepsilon|$ where ε is a specified very small number called tolerance.

Problem Statement:
Find the root of the equation

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65\tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2} = 0 \qquad \text{(E1)}$$

using the bisection method with $x_l^{(1)} = 0.0$, $x_u^{(1)} = 2.0$ and $\varepsilon = 0.005$.

Solution:
Modify the Fortran code, `ex1s31.f`, and Matlab code , `bisect.m`, to solve Eq. (E1) above.

- Function $f(x)$ is expressed using Taylor's series about an arbitrary point, x_1

$$f(x) = f(x_1) + (x - x_1)f'(x_1) + \frac{1}{2!}(x - x_1)^2 f''(x_1) + \dots \tag{3}$$

where $f, f', f'', \dots$ are evaluated at x_1

- Consider only the first two terms in the expansion

$$f(x) = f(x_1) + (x - x_1)f'(x_1) = 0 \tag{4}$$

and set $f(x) = 0$ to give

$$f(x_1) + (x - x_1)f'(x_1) = 0 \tag{5}$$

- Since higher order derivative terms were neglected in the approximation of $f(x)$ in Eq. (4), the solution of Eq. (5) yields the next approximation to the root (instead of the exact root) as

$$x = x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{6}$$

x_2 denotes improved approximation to the root. For the next improvement we use x_2 in place of x_1 on the RHS of Eq. (6) to obtain x_3

- The iterative procedure of Newton-Raphson method is generalized as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} : i = 1, 2, \ldots \tag{7}$$

Problem Statement:

Find the root of the equation

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65\tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2} = 0 \tag{E1}$$

using the Newton-Raphson method with the starting point $x_1 = 0.0$ and the convergence criteria $\varepsilon = 10^{-5}$.

Solution:

Modify the Fortran code, `ex1s32.f`, and Matlab code, `newton.m`, to solve Eq. (E1) above.

- Similar to Newton's method but is different in that the derivative f' is approximated using two consecutive iterative values of f

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \tag{8}$$

and Eq. (7) can be re-written as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - f(x_i)\left[\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}\right] \tag{9}$$

- Also known as Successive Substitution Method, in which function $f(x) = 0$ is re-written in the form of $x = g(x)$ and iterative procedure is used on

$$x_{i+1} = g(x_i) : i = 1, 2, 3, \ldots \tag{10}$$

where new approximation to root, x_{i+1}, is found using the previous one, x_i.

- Iterative process is stopped with a convergence criterion $10^{-3} < \varepsilon < 10^{-6}$

$$|x_{i+1} - g(x_{i+1})| \leq \varepsilon \tag{11}$$

- This method is simple but may NOT always converge with an arbitrarily chosen form of the function $g(x)$ but condition to be satisfied for convergence to the correct root is given by

$$|g'(x_{i+1})| < 1 \tag{12}$$

Problem Statement:
Find the root of the equation

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65\tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2} = 0 \qquad \text{(E1)}$$

using the fixed point iteration with the starting point $x_1 = 0.0$ and the convergence criteria $\varepsilon = 10^{-5}$.

Solution:
Re-arrange Eq. (E1)

$$\frac{1.5x}{(1+x^2)^2} = 0.65\tan^{-1}\left(\frac{1}{x}\right) - \frac{0.65x}{1+x^2} = 0$$

$$x = \underbrace{\frac{13}{30}(1+x^2)^2\tan^{-1}\left(\frac{1}{x}\right) - \frac{13}{30}x(1+x^2)}_{g(x)} \qquad \text{(E2)}$$

so the RHS is $g(x)$. Thus, from Eq. (E2), the iterative process can be expressed as

$$x_{i+1} = \frac{13}{30}(1+x_i^2)^2\tan^{-1}\left(\frac{1}{x_i}\right) - \frac{13}{30}x_i(1+x_i^2)$$

Problem Statement:
Find the root of the equation

$$f(x) = \tan^{-1} x = 0$$

using Matlab.

Solution:

Matlab Code

```
x = -2*pi:0.01:2*pi;
fx = 'atan(x)';
fplot(fx,[-2*pi 2*pi])
xlabel('x'),ylabel('y=f(x)'),title('Root Finding')
root = fzero(fx,1.6)
```

Problem Statement:

Find the root of the equation

$$f(x) = \sin 2x = 0$$

using Matlab.

Solution:

Matlab Code

```
x = -2*pi:0.01:2*pi;
fx = 'sin(2*x)';
fplot(fx,[-2*pi 2*pi])
xlabel('x'),ylabel('y=f(x)'),title('Root Finding')
root = fzero(fx,0.75)
```

Problem Statement:

Find the root of the equation

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65\tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2} = 0$$

using Matlab.

Solution:

Matlab Code

```
x = -2.0:0.01:2;
fx = '1.5*x./(1+x.^2).^2-0.65*(atan(1./x))+0.65*x./(1+x.^2)';
fplot(fx,[-2 2])
xlabel('x'),ylabel('y=f(x)'),title('Root Finding')
x2 = fzero(fx,-0.1)
```

Roots of Nonlinear Polynomials

- There is theorem which states that there is no such formula for general polynomials of degree higher than four.
- In practice we use numerical method to solve polynomial equations of degree higher than two.
- As a special case of $f(x) = 0$, consider a polynomial equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0 \tag{13}$$

where n denote degree of polynomial, a_0, a_1, $a_2 \ldots a_n$ are coefficients of polynomial. Eq. (13) in general will have n roots

- If $x_1, x_2 \ldots x_n$ are roots of Eq. (13), they are related to the coefficients of polynomial as

$$
\begin{aligned}
\sum_{i=1}^{n} x_i &= -\frac{a_{n-1}}{a_n} \\
\sum_{i=1}^{n} x_i \sum_{j=1, j\neq 1}^{n} x_i x_j &= -\frac{a_{n-2}}{a_n} \\
\sum_{i=1}^{n} x_i \sum_{j=1, j\neq i}^{n} \sum_{k=1, k\neq j}^{n} x_i x_j x_k &= -\frac{a_{n-3}}{a_n} \\
&\cdots \\
x_1 x_2 \ldots x_{n-1} x_n &= (-1)^n \frac{a_0}{a_n}
\end{aligned}
\tag{14}
$$

Your homework!

Read Section 2.11, pp. 84–88 of SINGIRESU S. RAO (2002): *Applied Numerical Methods for Engineers and Scientists*, ISBN 0-13-089480-X, Prentice Hall

Problem Statement:

Use Matlab to determine the roots of the following polynomial

$$f(x) = x^4 - 12x^3 + 0x^2 + 25x + 116 = 0$$

Solution:

Matlab Code

```
p = [1 -12 0 25 116]     % Assign coefficients to matrix p
r = roots(p)             % Find roots and assign them to r
pp = poly(r)             % Should return pp = p. This is a check

% You might want to plot the polynomial first
x = linspace(0,12);      % Set range for x
y = polyval(p,x);        % Evaluate polynomial at all values of x
xaxis = x*0;             % X-axis
plot(x,xaxis,x,y)        % Plot
```

Problem Statement:
Use Matlab to determine the roots of the following polynomial

$$f(x) = x^5 - 3.5x^4 + 2.75x^3 + 2.125x^2 - 3.875x + 1.25$$

which has three real roots: 0.5, -1.0 and 2.0, and a pair of complex roots: $-1 \pm 0.5i$.

Solution:

Matlab Code

```
%% Enter coefficients of polynomial into a matrix
p = [1 -3.5 2.75 2.125 -3.875 1.25]
%% You may evaluate polynomial f(x),say at x=1 or differentiate it...
polyval(p,1)
polyder(p)
%% Create quadratic eqn from two known roots:
%%   (x-0.5)(x+1) = x^2 + 0.5x - 0.5
r = [0.5 -1.0]; q = poly(r)
```

Matlab Code (continued)

```
% Divide the original polynomial f(x) by this quadratic
% eqn and assign results into quotient d and remainder e
[d e] = deconv(p,q)
roots(d)           % Roots of quotient polynomial d
conv(d,q)          % Multiply d by q to come up with the original f(x)

% Or, ALL roots may be determined by
r = roots(p)

% As a guide to where some of its roots might be, you may
% want to plot polynomial f(x) over a range, say, -10<x<10
x = [-10:0.1:10];

% Then set y = f(x) and x-axis ... and plot
y = x.^4-3.5*x.^3+2.75*x.^2-3.875*x+1.25;
yis0 = x*0;
plot(x,yis0,x,y)
```

Bibliography

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